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# Constant-sign Lyapunov functionals in the problem of the stability of a functional differential equation☆

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**Abstract**

The stability of the zero solution of a non-autonomous functional differential equation of the delayed type is investigated by means of limiting equations and a constant-sign Lyapunov functional, which has a constant-sign derivative. Special cases when the Lyapunov functional and its derivative are explicitly independent of time and the case of an almost periodic equation are also considered. The problem of stabilizing a pendulum in the upper unstable position and the problem of stabilizing the rotational motion of a rigid body are solved as examples.


## 1. Limiting equations

Suppose $R^p$ is a real linear space of $p$-vectors $\mathbf{x}$, $\mathbf{x}' = (x_1, x_2, \ldots, x_p)$ with a norm $|\mathbf{x}|$ (a prime denotes transposition), $h > 0$ is a real number, $C$ is the Banach space of continuous functions $\boldsymbol{\varphi}$: $[-h, 0] \to R^p$ with a norm $||\boldsymbol{\varphi}|| = \sup(|\boldsymbol{\varphi}(s)|, -h \le s \le 0)$ and, for $H > 0$,

$$C_H = \{\boldsymbol{\varphi} \in C : \|\boldsymbol{\varphi}\| < H\}$$

If $\mathbf{x}$: $R \to R^p$ is a continuous function, then, for $t \in R$, the function $\mathbf{x}_t \in C$ is defined by the equality

$$\mathbf{x}_t(s) = \mathbf{x}(t+s), \quad -h \le s \le 0$$

A right-hand derivative is denoted by $\dot{\mathbf{x}}(t)$

The functional differential equation with a finite delay

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}_t), \quad \mathbf{f}(t, \mathbf{0}) \equiv \mathbf{0} \tag{1.1}$$

is considered where $\mathbf{f}$: $R^+ \times C_H \to R^p$ is a continuous function which satisfies the conditions:[1,2]

a) it is bounded in each set $R^+ \times \bar{C}_L$, $\bar{C}_L = \{\boldsymbol{\varphi} : ||\boldsymbol{\varphi}|| \le L < H\}$:

$$|\mathbf{f}(t, \boldsymbol{\varphi})| \le m(L), \quad \forall (t, \boldsymbol{\varphi}) \in R^+ \times \bar{C}_L$$

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b) it satisfies the Lifschits condition with respect to $\boldsymbol{\varphi}$ in each compact set $K \subset C_H$:

$$|\mathbf{f}(t, \boldsymbol{\varphi}_2) - \mathbf{f}(t, \boldsymbol{\varphi}_1)| \le l(K)\|\boldsymbol{\varphi}_2 - \boldsymbol{\varphi}_1\|, \quad \forall t \in R^+, \quad \forall \boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2 \in K$$

c) it is uniformly continuous in each set $R^+ \times K$, where $K$ is an arbitrary compact set from $C_H$, i.e., for any $\varepsilon > 0$, a $\delta = \delta(\varepsilon, K) > 0$ exists such that

$$|\mathbf{f}(t_2, \boldsymbol{\varphi}_2) - \mathbf{f}(t_1, \boldsymbol{\varphi}_1)| \le \varepsilon, \quad \forall (t_1, \boldsymbol{\varphi}_1), (t_2, \boldsymbol{\varphi}_2) \in R^+ \times K, \quad |t_2 - t_1| \le \delta, \quad \|\boldsymbol{\varphi}_2 - \boldsymbol{\varphi}_1\| \le \delta$$

The smoothing of the solutions of Eq. (1.1) as $t$ increases follows from the first condition and, in particular, if $\mathbf{x} = \mathbf{x}(t, \alpha, \boldsymbol{\varphi})$ is a solution of Eq. (1.1) which satisfies the initial condition $\mathbf{x}_\alpha = \boldsymbol{\varphi}$ then, for the values $t \ge \alpha + h$, the function $\mathbf{x}_t \in \Gamma$, where $\Gamma \subset C_H$ is the union of the family of imbedded compact sets:[2]

$$\Gamma = \bigcup_{n=1}^{\infty} K_n, \quad K_1 \subset K_2 \subset \ldots \subset K_n \subset \ldots$$

The uniqueness of the solution $\mathbf{x} = \mathbf{x}(t, \alpha, \boldsymbol{\varphi})$ of Eq. (1.1) which satisfies the initial condition $\mathbf{x}_\alpha = \boldsymbol{\varphi}$ follows from the second condition and, moreover, this solution is defined in the maximum interval $[\alpha - h, \beta]$ and, if $\beta < \infty$, then $||\mathbf{x}_t(\alpha, \boldsymbol{\varphi})|| \to H$ when $t \to \infty$,[2] where

$$\mathbf{x}_t(\alpha, \boldsymbol{\varphi}) = \mathbf{x}(t + s, \alpha, \boldsymbol{\varphi}), \quad -h \le s \le 0$$

The precompactness of the family of shifts

$$\Phi = \{\mathbf{f}_\tau : \mathbf{f}_\tau(t, \boldsymbol{\varphi}) = \mathbf{f}(\tau + t, \boldsymbol{\varphi}), \tau \in R^+\}$$

in a certain space $F$ of continuous functions which are defined in the set $R \times \Gamma$ (Ref. 2) follows from the third condition.

Hence, a subsequence $\{t_{nk}\}$ and a function $\mathbf{f}^* \in F$ exist for each sequence $t_n \to \infty$ such that the sequence of functions $\mathbf{f}_k(t, \boldsymbol{\varphi}) = \mathbf{f}(t_{nk} + t, \boldsymbol{\varphi})$ converges in $F$ to the function $\mathbf{f}^*$. Denoting the family of limit functions by $\Phi^*$, we note that, if $\mathbf{f}^* \in \Phi^*$, then

$$\mathbf{f}^*(\tau + t, \boldsymbol{\varphi}) = \mathbf{f}^*_\tau \in \Phi^*, \quad \forall \tau \in R$$

The equation

$$\dot{\mathbf{x}}(t) = \mathbf{f}^*(t, \mathbf{x}_t), \quad \mathbf{f}^* \in \Phi^* \tag{1.2}$$

where the function $\mathbf{f}^* \in \Phi^*$ is defined in the set $R \times \Gamma$, is called the limiting equation to Eq. (1.1).

Note that a condition of the form of 2 is also satisfied in the case of the function $\mathbf{f}^*$, and the solution $\mathbf{x} = \mathbf{x}^*(t, \alpha, \boldsymbol{\varphi})$ of Eq. (1.2) is therefore unique for each initial point $(\alpha, \boldsymbol{\varphi}) \in R \times \Gamma$. Since $\mathbf{f}^*_\alpha \in \Phi^*$ for each $\alpha \in R$, we shall determine its solutions for each Eq. (1.2) when $\alpha = 0$, $\mathbf{x} = \mathbf{x}^*(t, \boldsymbol{\varphi}) = \mathbf{x}^*(t, 0, \boldsymbol{\varphi})$.

We shall make use of Theorem 1.1 from Ref. 1 (see also Ref. 2) in determining the correlation between the solutions of Eq. (1.1) and the solutions of Eq. (1.2).

Suppose $\mathbf{x} = \mathbf{x}(t, \alpha, \boldsymbol{\varphi})$ is a solution which is defined for all $t \ge \alpha - h$ and that $\omega^+(\alpha, \boldsymbol{\varphi})$ is its positive limit set in the space $C$: the point $\mathbf{p} \in \omega^+(\alpha, \boldsymbol{\varphi})$, if the sequence $t_n \to \infty$ exists, is such that $\mathbf{x}_t^{(n)}(\alpha, \boldsymbol{\varphi}) \to \mathbf{p}$ when $n \to \infty$, where

$$\mathbf{x}_t^{(n)}(\alpha, \boldsymbol{\varphi}) = \mathbf{x}(t_n + s, \alpha, \boldsymbol{\varphi}), \quad -h \le s \le 0$$

The following property of the quasi-invariance of the set $\omega^+(\alpha, \boldsymbol{\varphi})$ is determined by means of the limiting equations (1.2).[2] If $\mathbf{x} = \mathbf{x}(t, \alpha, \boldsymbol{\varphi})$ is a solution of Eq. (1.1) which is fixed and bounded:

$$|\mathbf{x}(t, \alpha, \boldsymbol{\varphi})| \le H_0 < H, \quad \forall t \ge \alpha - h$$

then, for each limit point $\mathbf{p} \in \omega^+(\alpha, \boldsymbol{\varphi})$, a limiting Eq. (1.2) exists such that the relation

$$\mathbf{x}^*_t(0, \mathbf{p}) \in \omega^+(\alpha, \boldsymbol{\varphi}), \quad \forall t \in R$$

is satisfied in the case of its solution $\mathbf{x}^*(t, 0, \mathbf{p})$.

This property, which is analogous to the property of the invariance of the positive limit set of the solution of an autonomous equation[3] enabled the use of Lyapunov functionals with a derivative of constant sign in problems concerned with the limiting behaviour of the solution of the non-autonomous Eq. (1.1),[4] to be validated.

## 2. The stability of the zero solution of Eq. (1.1)

We will now investigate a stability problem using constant-sign Lyapunov functionals.

Suppose $V: R^+ \times C_H \to R^+$ is a certain continuous functional and that $\mathbf{x} = \mathbf{x}(t, \alpha, \boldsymbol{\varphi})$ is a certain solution of Eq. (1.1). By means of the relation

$$\dot{V}(\alpha, \boldsymbol{\varphi}) = \limsup_{h \to 0+} \frac{1}{h}(V(\alpha + h, \mathbf{x}_{\alpha + h}(\alpha, \boldsymbol{\varphi})) - V(\alpha, \boldsymbol{\varphi}))$$

it is possible to determine the upper right-hand derivative of the functional $V$ at a point $(\alpha, \boldsymbol{\varphi})$.

We will assume that

$$\dot{V}(t, \boldsymbol{\varphi}) \le -W(t, \boldsymbol{\varphi}) \le 0, \quad \forall (t, \boldsymbol{\varphi}) \in R^+ \times C_0$$

$$C_0 = \{\boldsymbol{\varphi} \in C_H : \|\boldsymbol{\varphi}\| < H_0,\ 0 < H_0 < H\}$$

where $W: R^+ \times C_0 \to R^+$ is a continuous functional which is bounded and uniformly continuous in the set $R^+ \times K$ for each compact set $K \subset C_0$.

Like the family $\Phi$, the family of shifts

$$\Phi_W = \{W_\tau(t, \boldsymbol{\varphi}) = W(\tau + t, \boldsymbol{\varphi})\}$$

will be precompact in a certain space $F_W$ of the functionals

$$W : R \times \Gamma_0 \to R^+, \quad \Gamma_0 = C_0 \cap \Gamma$$

Hence, it is possible to determine the family of functions which are limiting to $W$ $\{W^*: R \times \Gamma_0 \to R^+\}$ and the family of limiting pairs $(\mathbf{f}^*, W^*)$.[4]

The following definitions will be important later.

**Definition 2.1.** Suppose $(\mathbf{f}^*, W^*)$ is the limiting pair which is determined by the sequence $t_n \to +\infty$. We define the set $V_\infty^{-1}(t, c)$, which is specified by the functional $V$ for each $t \in R$ and each $c \in R$, as the set of points for each of which a sequence $\{\boldsymbol{\varphi}_n \in C_0: \boldsymbol{\varphi}_n \to \boldsymbol{\varphi}\}$ exists such that, when $n \to \infty$, $V(t_n + t, \boldsymbol{\varphi}_n) \to c$. In particular,

$$V_\infty^{-1}(t, 0) = \{\boldsymbol{\varphi} \in \Gamma_0 : \exists \boldsymbol{\varphi}_n \to \boldsymbol{\varphi},\ V(t_n + t, \boldsymbol{\varphi}_n) \to 0,\ n \to \infty\}$$

**Definition 2.2.** The zero solution $\mathbf{x} = 0$ is stable with respect to the given limiting pair $(\mathbf{f}^*, W^*)$ and the corresponding set $V_\infty^{-1}(t, 0)$ if, for any $\varepsilon > 0$, a $\delta = \delta(\varepsilon) > 0$ exists such that, for any solution

$$\mathbf{x} = \mathbf{x}^*(t, \boldsymbol{\varphi}), \quad \mathbf{x}_0^* = \boldsymbol{\varphi}, \quad \boldsymbol{\varphi} \in \{\|\boldsymbol{\varphi}\| < \delta\} \cap V_\infty^{-1}(0, 0) \cap \{W^*(0, \boldsymbol{\varphi}) = 0\}$$

of Eq. (1.2),

$$|\mathbf{x}^*(t, \boldsymbol{\varphi})| < \varepsilon, \quad \forall t \ge 0$$

**Definition 2.3.** The zero solution $\mathbf{x} = 0$ is uniformly stable with respect to the limiting family $\{(\mathbf{f}^*, W^*), V_\infty^{-1}(t, 0)\}$ if the number $\delta > 0$ in Definition 2.2 is independent of the choice of the limiting pair $(\mathbf{f}^*, W^*)$.

**Definition 2.4.** The zero solution $\mathbf{x} = 0$ is asymptotically stable with respect to the given limiting pair $(\mathbf{f}^*, W^*)$ and the corresponding set $V_\infty^{-1}(t, 0)$ if it is stable and a $\Delta > 0$ exists for which, for any $\varepsilon > 0$, a $T = T(\varepsilon) > 0$ exists such that, for any solution

$$\mathbf{x} = \mathbf{x}^*(t, \boldsymbol{\varphi}), \quad \mathbf{x}_0^* = \boldsymbol{\varphi}, \quad \boldsymbol{\varphi} \in \{\|\boldsymbol{\varphi}\| < \Delta\} \cap V_\infty^{-1}(0,0) \cap \{W^*(0, \boldsymbol{\varphi}) = 0\}$$

of Eq. (1.2),

$$|\mathbf{x}^*(t, \boldsymbol{\varphi})| < \varepsilon, \quad \forall t \geq T$$

**Definition 2.5.** The zero solution $\mathbf{x}=0$ is uniformly asymptotically stable with respect to the limiting family $\{(\mathbf{f}^*, W^*), V_\infty^{-1}(t, 0)\}$ if the numbers $\Delta>0$ and $T>0$ in Definition 2.4 are independent of the choice of the limiting pair $(\mathbf{f}^*, W^*)$.

The definitions which have been introduced enable us to derive the sufficient conditions of stability and asymptotic stability when a non-negative functional with a non-positive derivative exists.

**Theorem 2.1.** *We assume that*:

1) *a continuous functional*

$$V(t, \boldsymbol{\varphi}) \geq 0, \quad V(t, 0) \equiv 0$$

*exists in the domain* $R^+ \times C_0$ *with a derivative*

$$\dot{V}(t, \boldsymbol{\varphi}) \leq -W(t, \boldsymbol{\varphi}) \leq 0;$$

2) *the zero solution* $\mathbf{x}=0$ *is uniformly asymptotically stable with respect to the limit set* $\{(\mathbf{f}^*, W^*), V_\infty^{-1}(t, 0)\}$.

*The zero solution of* Eq. (1.1) *is then stable*.

**Proof.** We will assume that the solution of Eq. (1.1) $\mathbf{x}=0$ is unstable: for a certain $\alpha \in R^+$, a $\varepsilon_0>0$ and sequences $\{\boldsymbol{\varphi}_n\colon \|\boldsymbol{\varphi}_n\| \to 0\}$ and $\beta_n \to \infty$ exist when $n \to \infty$ such that

$$|\mathbf{x}(\alpha + \beta_n, \alpha, \boldsymbol{\varphi}_n)| \geq \varepsilon_0 \tag{2.1}$$

Suppose $\Delta>0$ is a number which is determined from Condition 2 of the theorem. We put

$$2\eta = \min(\Delta, \varepsilon_0)$$

It follows from inequality (2.1) that a sequence of values $\gamma_n \to \infty$ exists such that

$$|\mathbf{x}(\alpha + t, \alpha, \boldsymbol{\varphi}_n)| < \eta \ (-h \leq t < \gamma_n), \quad |\mathbf{x}(\alpha + \gamma_n, \alpha, \boldsymbol{\varphi}_n)| = \eta \tag{2.2}$$

It follows from the continuity of the functional $V=V(t, \boldsymbol{\varphi})$ at the point $\boldsymbol{\varphi}=0$ that

$$V_0^{(n)} = V(\alpha, \boldsymbol{\varphi}_n) \to 0, \quad n \to \infty$$

and, moreover, it can be assumed that the sequence $V_0^{(n)}$ is a monotonically decreasing sequence. From Condition 1 of the theorem, for each solution $\mathbf{x}=\mathbf{x}(t, \alpha, \boldsymbol{\varphi}_n)$ we derive, for each $t \geq 0$, the relation

$$\begin{gathered} 0 \leq V^{(n)}[s+t] \leq V^{(n)}[s-t] - \int_{s-t}^{s+t} W^{(n)}[\tau]d\tau \leq V_0^{(n)} \\ V^{(n)}[\tau] = V(\tau, \mathbf{x}_\tau(\alpha, \boldsymbol{\varphi}_n)), \quad W^{(n)}[\tau] = W(\tau, \mathbf{x}_\tau(\alpha, \boldsymbol{\varphi}_n)) \end{gathered} \tag{2.3}$$

Suppose $T=T(\eta/2)>0$ is a number which is determined from Condition 2 of the theorem in accordance with Definition 2.5. For any limiting Eq. (1.2), each solution of it $\mathbf{x}=\mathbf{x}^*(t, \boldsymbol{\psi})$ is such that

$$\boldsymbol{\psi} \in \{\|\boldsymbol{\varphi}\| < \Delta\} \cap V_\infty^{-1}(0,0) \cap \{W^*(0, \boldsymbol{\varphi}) = 0\}$$

satisfies the inequality

$$|\mathbf{x}^*(t, \boldsymbol{\psi})| < \eta/2, \quad \forall t \geq T \tag{2.4}$$

We put $\mu_n = \alpha + \gamma_n - T$ and make up the sequence

$$\{\boldsymbol{\psi}_n \in C : \boldsymbol{\psi}_n = \mathbf{x}(\mu_n + s, \alpha, \boldsymbol{\varphi}_n), -h \leq s \leq 0\}$$

On taking account of the fact that

$$\mathbf{x}(\mu_n + t, \alpha, \boldsymbol{\varphi}_n) = \mathbf{x}(\mu_n + t, \mu_n, \boldsymbol{\psi}_n)$$

from relations (2.2) we have

$$\|\boldsymbol{\psi}_n\| < \eta, \quad |\mathbf{x}(\mu_n + T, \mu_n, \boldsymbol{\psi}_n)| = \eta \tag{2.5}$$

Inequalities (2.3) are easily reduced to the form

$$0 \leq V^{(n)}[\mu_n + t] \leq V^{(n)}[\mu_n - t] - \int_{-t}^{t} W^{(n)}[\mu_n + \tau] d\tau \leq V_0^{(n)}$$
$$V^{(n)}[\tau] = V^{(n)}(\tau, \mathbf{x}_\tau(\mu_n, \boldsymbol{\psi}_n)), \quad W^{(n)}[s] = W(s, \mathbf{x}_s(\mu_n, \boldsymbol{\psi}_n)) \tag{2.6}$$

From the sequence $\{\mu_n\}$, we select the subsequence $\{\mu_{nk}\}$ for which

$$\boldsymbol{\psi}_{nk} \to \boldsymbol{\psi}^* \text{ in } C, \mathbf{f}(\mu_{nk} + t, \boldsymbol{\varphi}) \to \mathbf{f}^*(t, \boldsymbol{\varphi}) \text{ in } F, W(\mu_{nk} + t, \boldsymbol{\varphi}) \to W^*(t, \boldsymbol{\varphi}) \text{ in } F_W$$

According to Theorem (1.1) from Ref. 1, the sequence $\{\mathbf{x}(\mu_{nk} + t, \mu_{nk}, \boldsymbol{\psi}_{nk})\}$ will converge uniformly with respect to $t \in [-\beta, \beta]$ for each $\beta \geq 0$ to the solution $\mathbf{x} = \mathbf{x}^*(t, \boldsymbol{\psi}^*)$ of Eq. (1.2). For this solution, we obtain from relations (2.5)

$$\|\boldsymbol{\psi}\| \leq \eta < \Delta, \quad |\mathbf{x}^*(T, 0, \boldsymbol{\psi}^*)| = \eta \tag{2.7}$$

On taking the limit as $n \to \infty$, we find from inequalities (2.6) that

$$\mathbf{x}_t^*(0, \boldsymbol{\psi}^*) \in V_\infty^{-1}(t, 0) \cap \{W^*(t, \boldsymbol{\varphi}) = 0\}, \forall t \in R$$

and this, according to relations (2.7), contradicts the choice of the number $T$. □

The following theorem, in which a continuous, strictly monotonically decreasing function is denoted by $a: R^+ \to R^+$, is proved in a similar manner.

**Theorem 2.2.** *If, in addition to the conditions of* Theorem (2.1), *the estimate* $V(t, \boldsymbol{\varphi}) \leq a(||\boldsymbol{\varphi}||)$ *when* $(t, \boldsymbol{\varphi}) \in R^+ \times C_0$ *holds for the functional* $V(t, \boldsymbol{\varphi})$, *then the zero solution of* Eq. (1.1) *is uniformly stable.*

Note that, subject to the conditions of Theorems 2.1 and 2.2, the set $V_\infty^{-1}(t, 0) \cap \{W^*(t, \boldsymbol{\varphi}) = 0\}$ is invariant with respect to Eq. (1.2) for each limiting pair $(\mathbf{f}^*, W^*)$.

Correspondingly, we denote the set

$$M(t, c_0) = \{V_\infty^{-1}(t, c) : c = c_0\} \cap \{W^*(t, \boldsymbol{\varphi}) = 0\} \tag{2.8}$$

by $M(t, c_0)$ and the subset of the set $M(t, c_0)$ that is the largest subset which is invariant with respect to the limiting Eq. (1.2) by $N(c_0)$.

**Theorem 2.3.** *We assume that Conditions 1 and 2 of* Theorem 2.1 *are satisfied and, also, the following condition: for each limiting pair* $(\mathbf{f}^*, W^*)$ *and for each small value* $c = c_0 = \text{const} \geq 0$, *the set*

$$N(c_0) \subset N(0)$$

*The zero solution of Eq.* (1.1) *is then asymptotically stable.*

**Theorem 2.4.** *We assume that the conditions of* Theorem 2.1 *are satisfied and, also, the following condition: a limiting pair* $(\mathbf{f}^*, W^*)$ *exists such that, for any small* $c = c_0 > 0$, *the set* (2.8) *does not contain solutions of the limiting* Eq. (1.2).

*The zero solution of Eq.* (1.1) *is then asymptotically stable uniformly with respect to* $\boldsymbol{\varphi}$.

*The correctness of* Theorems 2.3 and 2.4 *follows from* Theorem 2.1 *and* Theorems 2.1 and 3.4 *in* Ref. 4.

**Theorem 2.5.** *We assume that the conditions of* Theorem 2.2 *are satisfied and, also, the following condition: for each limiting pair* $(\mathbf{f}^*, W^*)$ *and for each small* $c = c_0 > 0$, *the set* (2.8) *does not contain solutions of the limiting* Eq. (1.2).

*The zero solution of* Eq. (1.1) *is then uniformly asymptotically stable*.

**Proof.** According to Theorem 2.2, the solution $\mathbf{x} = 0$ of Eq. (1.1) is uniformly stable.

Suppose $\Delta$ is a number which is determined from Condition 2 of Theorem 2.1 according to Definition 2.5. We now determine the domain $\bar{C}_{H_0}(0 < H_0 \leq \Delta_1 = \Delta/2)$, the solutions from which are uniformly bounded by the domain $\bar{C}_{\Delta_1}$:

$$|\mathbf{x}(t, \alpha, \boldsymbol{\varphi})| \leq \Delta_1 = \Delta/2, \quad \forall(\alpha, \boldsymbol{\varphi}) \in R^+ \times \bar{C}_{H_0}, \quad \forall t \geq \alpha - h \tag{2.9}$$

To prove the theorem, it is sufficient to show that $\mathbf{x} = 0$ is the point of uniform attraction of the solutions

$$\mathbf{x} = \mathbf{x}(t, \alpha, \boldsymbol{\varphi}), \quad (\alpha, \boldsymbol{\varphi}) \in R^+ \times \bar{C}_{H_0}$$

We shall assume the opposite: a number $\varepsilon_0 > 0$ exists for which, in the case of any sequence $\sigma_n \to +\infty$, a sequence of initial values $(\alpha_n, \boldsymbol{\varphi}_n) \in R^+ \times \bar{C}_{H_0}$ exists such that the solutions $\mathbf{x} = \mathbf{x}(t, \alpha_n, \boldsymbol{\varphi}_n)$ satisfy the inequality

$$|\mathbf{x}(\alpha_n + \sigma_n, \alpha_n, \boldsymbol{\varphi}_n)| \geq \varepsilon_0. \tag{2.10}$$

For given $\varepsilon_0 > 0$, we determine the number $\delta = \delta(\varepsilon_0) > 0$ using the property of uniform stability. It then follows from inequalities (2.9) and (2.10) that

$$\|\mathbf{x}_t(\alpha_n, \boldsymbol{\varphi}_n)\| \leq \Delta_1, \quad \|\mathbf{x}_t(\alpha_n, \boldsymbol{\varphi}_n)\| \geq \delta, \quad \forall t \geq \alpha_n \tag{2.11}$$

Along each solution $\mathbf{x} = \mathbf{x}(t, \alpha_n, \boldsymbol{\varphi}_n)$, the function $V^{(n)}[t] = V(t, \mathbf{x}_t(\alpha_n, \boldsymbol{\varphi}_n))$ $(t \geq \alpha_n)$ is monotonically decreasing with respect to $t$ and, at the same time, $V^{(n)}[t] \searrow 0$ when $t \to +\infty$ (see Ref. 4).

For a certain sequence $\{c_0^{(n)}\}$, $c_0^{(n)} \searrow 0$ when $n \to \infty$, we choose a sequence $\{\mu_n\}$ such that

$$V_0^{(n)} = V^{(n)}[\mu_n] = c_0^{(n)}$$

Then,

$$\lim_{n \to \infty} V^{(n)}[\mu_n + t] = 0, \quad \forall t \in R \tag{2.12}$$

We form the sequences

$$\{\boldsymbol{\psi}_n = \mathbf{x}_{\mu_n}(\alpha_n, \boldsymbol{\varphi}_n)\}, \{\mathbf{f}_n = \mathbf{f}(\mu_n + t, \boldsymbol{\varphi})\}, \{W_n = W(\mu_n + t, \boldsymbol{\varphi})\}$$

and choose the subsequences for which

$$\boldsymbol{\psi}_{n_k} \to \boldsymbol{\varphi}^*, \ \mathbf{f}_{n_k} \to \mathbf{g}^* \ \text{ in } F, \ W_{n_k} \to U^* \ \text{ in } F_W, \text{ when } n_k \to \infty$$

According to Theorem 1.1 from Ref. 1, the sequence of functions

$$\mathbf{x}^{(n_k)}(t) = \mathbf{x}(\mu_{n_k} + t, \alpha_{n_k}, \boldsymbol{\varphi}_{n_k})$$

will converge to the solution $\mathbf{x} = \mathbf{x}^*(t, 0, \boldsymbol{\varphi}^*)$ of the equation $\dot{\mathbf{x}}(t) = \mathbf{g}^*(t, \mathbf{x}_t)$. At the same time, it follows from the estimates (2.11) that

$$\|\boldsymbol{\varphi}^*\| \leq \Delta_1, \quad \|\mathbf{x}_t^*(0, \boldsymbol{\varphi}^*)\| \geq \delta, \quad \forall t \geq 0 \tag{2.13}$$

Taking the limit as $n_k \to +\infty$, we obtain from relations of the form of (2.6), when account is taken of relation (2.12), that

$$\mathbf{x}_t^*(0, \boldsymbol{\varphi}^*) \in V_\infty^{-1}(t, 0) \cap \{U^*(t, \boldsymbol{\varphi}) = 0\}$$

But this, together with estimates (2.13), contradicts the last condition of the theorem. □

## 3. The stability of an almost periodic equation

Suppose the right-hand side of Eq. (1.1) is defined in the set $R \times C_H$ and is a continuous and almost periodic function of $t$ according to the following definition.

**Definition 3.1.** The function $\mathbf{f}$: $R \times C_H \to R^p$ is said to be almost periodic in $t$ if, for each compact set $K \subset C_H$ and each number $\varepsilon > 0$, a number $l = l(K, \varepsilon) > 0$ exists such that, for each $b \in R$, a number $T \in [b, b + l]$ is found such that

$$|\mathbf{f}(t+T, \boldsymbol{\varphi}) - \mathbf{f}(t, \boldsymbol{\varphi})| < \varepsilon, \quad \forall (t, \boldsymbol{\varphi}) \in R \times K$$

The number $T$ is called the $\varepsilon$-almost period of the function $\mathbf{f}$.

The continuous, almost periodic function $\mathbf{f}$ is uniformly continuous in the set $R \times K$ and the construction of the limiting equations therefore has the following special features. The family of shifts

$$\Phi = \{\mathbf{f}_\tau(t, \boldsymbol{\varphi}) = \mathbf{f}(\tau + t, \boldsymbol{\varphi}), \tau \in R\}$$

is precompact in the space $F$ of functions $\mathbf{f}$: $R \times \Gamma \to R$ which are almost periodic in $t$ with a metric

$$d(\mathbf{f}, \mathbf{g}) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(\mathbf{f}, \mathbf{g})}{1 + d_n(\mathbf{f}, \mathbf{g})}, \quad d_n(\mathbf{f}, \mathbf{g}) = \sup(|\mathbf{f} - \mathbf{g}|, (t, \boldsymbol{\varphi}) \in R \times K_n) \tag{3.1}$$

where $K_n \subset \Gamma$ are the compact sets mentioned in Section 1. This metric is stronger compared with the metric introduced earlier in Ref. 2 for the general case of the non-autonomous Eq. (1.1).

It is easy to derive the following property of an almost periodic function.

**Lemma 3.1.** *In the case of a function* $\mathbf{f}$: $R \times C_H \to R^p$ *which is almost periodic in t, a limiting function* $\mathbf{f}^* \in \Phi^*$ *exists which is identical to it in the set* $R \times \Gamma$.

*Therefore, at least one of the limiting Eq. (1.2) is identical to the initial Eq. (1.1).*

We will assume that the functional $V(t, \varphi)$ is continuous and almost periodic in $t$ in accordance with Definition 3.1. The family of shifts of this functional

$$\Phi_V = \{V_\tau(t, \boldsymbol{\varphi}) = V(t + \tau, \boldsymbol{\varphi}), \tau \in R\}$$

will be precompact in the space $F_V$ of almost periodic functionals $V$: $R \times \Gamma \to R^+$ with a metric similar to (3.1). Consequently, a family $\Phi_V^*$ of functionals $V^*$ which are limiting to $V$ is defined and, among these, there is also a functional which is identical to $V$ for which the set

$$V_\infty^{-1}(t, c) = \{\boldsymbol{\varphi} \in \Gamma : V^*(t, \boldsymbol{\varphi}) = c\}$$

From these constructions and Theorem 2.4, we have the following result.

**Theorem 3.1.** *Suppose* Eq. (1.1) *is almost periodic and such that*:

1) *a functional* $V$: $R \times C_0 \to R^+$ *exists which has a derivative with an estimate* $\dot{V}(t, \varphi) \leq 0$;
2) *the zero solution* $\mathbf{x} = 0$ *is uniformly asymptotically stable with respect to the set* $(\mathbf{f}^*, \{V^*(t, \varphi) = 0\})$;
3) *the set* $\{V(t, \varphi) > 0\} \cap \{V(t, \varphi) = 0\}$ *does not contain solutions of* Eq. (1.1).

*The zero solution of* Eq. (1.1) *is then asymptotically stable uniformly with respect to* $\varphi$.

## 4. Modification of Theorems 2.1 to 2.5

In the special case when the functionals $V$ and $W$ are not explicitly time-dependent, the conditions of Theorems 2.1–2.5 can be reduced to the corresponding conditions with respect to the initial Eq. (1.1) by introducing the following definitions.

**Definition 4.1.** The zero solution of Eq. (1.1) is uniformly stable with respect to the set $\{V(\boldsymbol{\varphi})=0\}$ if, for any $\varepsilon>0$, a $\delta=\delta(\varepsilon)>0$ exists such that, for all $\boldsymbol{\varphi}\in\{\|\boldsymbol{\varphi}\|<\delta\}\cap\{V(\boldsymbol{\varphi})=0\}$:

$$|\mathbf{x}(t,\alpha,\boldsymbol{\varphi})|<\varepsilon,\quad \forall\alpha\geq 0,\quad \forall t\geq\alpha$$

**Definition 4.2.** The zero solution of Eq. (1.1) is uniformly asymptotically stable with respect to the set $\{V(\boldsymbol{\varphi})=0\}$ if it is uniformly stable with respect to this set and a $\Delta>0$ also exists for which, for any $\varepsilon>0$, a $T=T(\varepsilon)>0$ exists such that, for all $\boldsymbol{\varphi}\in\{\|\boldsymbol{\varphi}\|<\Delta\}\cap\{V(\boldsymbol{\varphi})=0\}$:

$$|\mathbf{x}(t,\alpha,\boldsymbol{\varphi})|<\varepsilon,\quad \forall\alpha\geq 0,\quad \forall t\geq\alpha+T$$

It can be shown in a similar manner to the derivation of Theorem 2.1 in Ref. 2 that, if the zero solution of Eq. (1.1) is uniformly asymptotically stable with respect to the set $\{V(\boldsymbol{\varphi})=0\}$, then the solution $\mathbf{x}=0$ is uniformly asymptotically stable with respect to the limiting family $((\mathbf{f}^*, W^*), V^{-1}(0))$. Correspondingly, we have the following result from Theorem 2.2.

**Theorem 4.1.** *We assume that*:

1) *a functional $V(\boldsymbol{\varphi})\geq 0$ with a derivative $\dot{V}(t,\boldsymbol{\varphi})\leq 0$ exists in the domain $C_0$*;
2) *the zero solution of* Eq. (1.1) *is uniformly asymptotically stable with respect to the set* $\{V(\boldsymbol{\varphi})=0\}$.

*It is then uniformly stable*.

**Definition 4.3.** The set

$$M(c_0)=\{V(\boldsymbol{\varphi})=c_0>0\}\cap\{W(\boldsymbol{\varphi})=0\} \tag{4.1}$$

does not contain solutions of Eq. (1.1) uniformly with respect to $\alpha\in R^+$ if a $T=T(c_0)>0$ exists such that, for any $\boldsymbol{\varphi}\in M(c_0)$,

$$\mathbf{x}_t(\alpha,\boldsymbol{\varphi})\notin M(c_0),\quad \forall t\geq\alpha+T$$

It can be shown that, if the initial Eq. (1.1) has the property corresponding to definition (4.3), then the last condition of Theorem 2.5 is satisfied for the family of limiting equations (1.2).

Correspondingly, the following theorem holds.

**Theorem 4.2.** *We assume that the conditions of* Theorem 4.1 *are satisfied and, also, the following condition: the set* (4.1) *does not contain solutions of* Eq. (1.1) *uniformly with respect to* $\alpha\in R^+$.

The zero solution of Eq. (1.1) is then uniformly asymptotically stable.

The advantage of the more particular Theorems 4.1 and 4.2 compared with Theorems 2.2 and 2.5 lies in the fact that they do not contain conditions on the family of limiting equations and limiting functionals, and this means that there is no need to carry out their corresponding construction.

On the basis of the enhancement of the definitions of stability of the solution $\mathbf{x}=0$ with respect to a given set, it is possible to derive results which are analogous to Theorems 2.1–2.5 with conditions only imposed on the initial Eq. (1.1) and the functionals $V(t,\boldsymbol{\varphi})$ and $W(t,\boldsymbol{\varphi})$. For example, it is possible, by introducing the following definitions, to obtain simpler sufficient conditions of asymptotic stability.

**Definition 4.4.** The zero solution of Eq. (1.1) is uniformly stable with respect to the set $\{V(t,\boldsymbol{\varphi})=0\}$ if, for any $\alpha\in R^+$ and for any $\varepsilon>0$, $\delta_1=\delta_1(\varepsilon)>0$, $\delta_2=\delta_2(\varepsilon)>0$ exist such that, for any $\boldsymbol{\varphi}\in\{\|\boldsymbol{\varphi}\|<\delta_1\}\cap\{V(\alpha,\boldsymbol{\varphi})<\delta_2\}$:

$$|x(t, \alpha, \boldsymbol{\varphi})| < \varepsilon, \quad \forall t \geq \alpha$$

The zero solution of Eq. (1.1) is uniformly asymptotically stable with respect to the set $\{V(t, \boldsymbol{\varphi}) = 0\}$ if it is uniformly stable with respect to this set and $\Delta_1 > 0$ $\Delta_2 > 0$ exist for which, for any $\alpha \in R^+$ and $\varepsilon > 0$, a $T = T(\varepsilon) > 0$ exists such that, for all $\boldsymbol{\varphi} \in \{||\boldsymbol{\varphi}|| < \Delta_1\} \cap \{V(\alpha, \boldsymbol{\varphi}) < \Delta_2\}$:

$$|\mathbf{x}(t, \alpha, \boldsymbol{\varphi})| < \varepsilon, \quad \forall t \geq \alpha + T$$

**Theorem 4.3.** *We assume that*:

1) *a continuous functional*

$$V(t, \boldsymbol{\varphi}) \geq 0, \quad V(t, 0) \equiv 0, \quad V(t, \boldsymbol{\varphi}) \leq a_1(\|\boldsymbol{\varphi}\|)$$

*exists in the domain* $R^+ \times C_0$ *and, in the domain* $\{V(t, \boldsymbol{\varphi}) = 0\}$, *its derivative satisfies the inequality*

$$\dot{V}(t, \boldsymbol{\varphi}) \leq -a_2(V(t, \boldsymbol{\varphi}))$$

2) *the zero solution of* Eq. (1.1) *is uniformly asymptotically stable with respect to the set* $\{V(t, \boldsymbol{\varphi}) = 0\}$.

*The solution is then uniformly asymptotically stable.*

The modification of Theorem 2.5, which provides global asymptotic stability of the zero solution of Eq. (1.1) with a domain of definition $R^+ \times C$, is important.

**Definition 4.5.** The solutions of Eq. (1.1) are uniformly bounded if, for any $\alpha \geq 0$ and any $H_0 > 0$, a $H_1 = H_1(H_0) > 0$ is found such that, for any $\boldsymbol{\varphi}$: $||\boldsymbol{\varphi}|| < H_0$:

$$|\mathbf{x}(t, \alpha, \boldsymbol{\varphi})| \leq H_1, \quad \forall t \geq \alpha$$

**Theorem 4.4.** *We assume that, in addition to the conditions of* Theorem 2.5, *the solutions of* Eq. (1.1) *are uniformly bounded*.

*The zero solution of* Eq. (1.1) *is then uniformly globally asymptotically stable*.

## 5. Examples

### *5.1. Example of the stabilization of a pendulum in the upper unstable position*

Consider the problem of the stabilization of a mathematical pendulum in the upper unstable equilibrium position by a moment applied to it on the axis of suspension in a linear formulation.

The linear approximation equations can be reduced to the form[5]

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_1 + x_3, \quad \dot{x}_3 = u \tag{5.1}$$

where $x_1 = \varphi$ is the angle of inclination of the pendulum from the vertical, $x_2 = \dot{\varphi}$ and $u$ is the control moment applied to the pendulum.

We will assume that, in the pendulum control system, the coordinates $x_1, x_2, x_3$ are determined with a delay $\tau = \tau(t)$, $0 \leq \tau(t) \leq h$ and that $\tau(t)$ is a function which is uniformly continuous in $t \in R^+$. The control moment

$$u = -4(x_1((t - \tau(t)) + x_2(t - \tau(t))) - 3x_3(t - \tau(t)) \tag{5.2}$$

is given

We define the Lyapunov functional by the equality

$$V(\varphi_1, \varphi_2, \varphi_3) = 4(\varphi_1(0) + \varphi_2(0))^2 + 4(\varphi_1(0) + \varphi_2(0))\varphi_3(0) + \frac{3}{2}\varphi_3^2(0) + $$
$$+\frac{1}{4h}\int_{-2h}^{0}\left(\int_{s}^{0}(4(\varphi_1(u) + \varphi_2(u))^2 + \varphi_3^2(u))du\right)ds$$

This functional is non-negative and, when $h \leq 1/18$, it has a derivative, by virtue of relations (5.1) and (5.2), which satisfies the inequality

$$\dot{V}(t, \varphi_1, \varphi_2, \varphi_3) \leq -W(\varphi_1, \varphi_2, \varphi_3) = -\int_{-2h}^{0}((\varphi_1(s) + \varphi_2(s))^2 + \varphi_3^2(0))ds \leq 0$$

The system, which is the limiting system to (5.1), will have the form

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_1 + x_3, \quad \dot{x}_3 = -4(x_1((t - \tau^*(t)) + x_2(t - \tau^*(t))) - 3x_3(t - \tau^*(t)) \tag{5.3}$$

It is obvious that the solution $x_1 = x_2 = x_3 = 0$ of system (5.3) is asymptotically stable with respect to the set

$$\{V = 0\} = \{(\varphi_1, \varphi_2, \varphi_3) : \varphi_1(s) + \varphi_2(s) = 0, \varphi_3(s) = 0, -h \leq s \leq 0\}$$

It follows from the estimate for the derivative of the functional that the set

$$\{W = 0\} = \{(\varphi_1, \varphi_2, \varphi_3) : \varphi_1(s) + \varphi_2(s) = 0, \varphi_3(0) = 0, -h \leq s \leq 0\}$$

The set $\{V = c = \text{const} > 0\} \cap \{W = 0\}$ does not contain solutions of system (5.3).

On the basis of Theorem 4.4, it can be asserted, in the case of the control moment (5.2), that the zero position of system (5.1) will be globally uniformly asymptotically stable.

This example was considered as a model example previously[6] and an algorithm for constructing a control with delayed feedback under the action of perturbations was presented. The solutions obtained above are the analytical solutions of the problem.

### 5.2. *The problem of stabilizing the rotational motion of a rigid body*

We will denote the moments of inertia about the principal $Ox$, $Oy$ and $Oz$ axes for a body with a fixed point $O$ by $A$, $B$ and $C$, and $p$, $q$ and $r$ are the projections of the angular velocity on to these axes. We shall assume that, under the action of a moment

$$M_x = M_y = 0, \quad M_z = M_z(t)$$

the body executes unsteady rotational motion of the form

$$p = q = 0, \quad r = r_0(t), \quad |r_0(t)| \leq R_0 = \text{const} > 0 \tag{5.4}$$

about the $Oz$ axis, where $r_0(t)$ is a function which is uniformly continuous in $t \in R^+$.

In order to solve the problem of stabilizing this motion by control moments $M_1$, $M_2$, $M_3$, we will construct the equations of the perturbed motion

$$\begin{aligned} A\dot{x}_1 &= (B - C)x_2(r_0 + x_3) + M_1 \\ B\dot{x}_2 &= (C - A)(r_0 + x_3)x_1 + M_2 \\ C\dot{x}_3 &= (A - B)x_1x_2 + M_3 \end{aligned} \tag{5.5}$$

where $x_1 = p$, $x_2 = q$, $x_3 = r - r_0$.

We will assume that, in the system controlling the body, the coordinates $x_1$, $x_2$ and $x_3$ are determined with delays $\tau_1(t)$, $\tau_2(t)$ and $\tau_3(t)$, $0 \leq \tau_k(t) \leq h > 0$ and that $\tau_i$, $i = 1, 2, 3$ are uniformly continuous functions when $t \in R^+$.

We separate the solution of the problem of stabilizing the motion (5.4) by the moments

$$M_1 = -b_1 x_1(t-\tau_1(t)), \quad M_2 = -b_1 x_2(t-\tau_2(t)), \quad M_3 = -b_2 x_3(t-\tau_3(t))$$

into two problems: the problem of stabilization with respect to $x_1$ and $x_2$ on the basis of the first two equations of system (5.5) and the problem of asymptotic stability with respect to $x_3$ on the basis of the third equation. These problems can be respectively solved in the case of the condition $A>B>C$ (or $A<B<C$) using the constant-sign functionals

$$V_1(\varphi_1, \varphi_2) = \frac{1}{2}A(A-C)\varphi_1^2(0) + \frac{1}{2}B(B-C)\varphi_2^2(0) + \mu_1 \int_{-2h}^{0} I_1(s)ds + \mu_2 \int_{-2h}^{0} I_2(s)ds$$

$$\mu_1 = \frac{A-C}{2B}[b_1 R_0(B-C) + b_1^2], \quad \mu_2 = \frac{B-C}{2B}[b_1 R_0(A-C) + b_1^2]$$

$$V_2(\varphi_3) = \frac{1}{2}C\varphi_3^2(0) + \frac{b_2^2}{2C}\int_{-2h}^{-h} I_3(s)ds$$

$$I_i(s) = \int_s^0 \varphi_i^2(\nu)d\nu, \quad i = 1, 2, 3$$

Suppose the amplification factors $b_1$ and $b_2$ satisfy the conditions

$$0 < b_1 < \frac{B}{2h} - 2R_0(A-C), \quad 0 < hb_2 < C \tag{5.6}$$

The system, which is the limiting system to (5.5), has the form

$$\begin{aligned} A\dot{x}_1 &= (B-C)x_2(r_0^* + x_3) - b_1 x_1(t-\tau_1^*(t)) \\ B\dot{x}_2 &= (C-A)(r_0^* + x_3)x_1 - b_1 x_2(t-\tau_2^*(t)) \\ C\dot{x}_3 &= (A-B)x_1 x_2 - b_2 x_3(t-\tau_3^*(t)) \end{aligned} \tag{5.7}$$

The derivative of the functional $V_1$ has the limit

$$\dot{V}_1(t, \varphi_1, \varphi_2) \le -W_1(\varphi_1, \varphi_2) =$$
$$= -\frac{(A-B)(A-C)}{2AB}[b_1^2 I_1(-2h) + b_1 R_0(B-C)I_2(-2h)] \le 0$$

The set

$$\{V_1 = c = \text{const} > 0\} \cap \{W_1 = 0\}$$

does not contain solutions of system (5.7). According to Theorem 2.5 from Ref. 1, we obtain that the zero solution of system (5.5) is uniformly asymptotically stable with respect to $x_1$ and $x_2$.

The derivative of the functional $V_2$ has the estimate

$$\dot{V}_2(t, \varphi_3) \le -W_2(\varphi_3) = -b_2\left(1 - \frac{b_2 h}{C}\right)\varphi_3^2(0) \le 0$$

Taking account of the fact that the zero solution of system (5.5) is uniformly asymptotically stable in $x_1$ and $x_2$, it can be asserted that the zero solution $\mathbf{x}=0$ of system (5.5) is uniformly asymptotically stable with respect to the set

$$\{V_2 = 0\} = \{\varphi_3(s) = 0, -h \ge s \ge 0\}$$

By virtue of the fact that

$$x_1(t) \to 0, \quad x_2(t) \to 0, \quad t \to +\infty$$

the limiting system (5.7) has the form

$$C\dot{x}_3 = -b_2 x_3(t - \tau_3^*(t)) \tag{5.8}$$

The set

$$\{V_2 = c = \text{const} > 0\} \cap \{W_2 = 0\}$$

does not contain solutions of system (5.8).

Using Theorem 4.2, we infer that the rotation (5.4) about the largest and smallest inertial axes under the action of moments (5.5) with amplification factors which satisfy conditions (5.6) is uniformly asymptotically stable.

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